A Unifying Framework for Longitudinal Flying Qualities Criteria

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The objective of this paper is to interpret existing flying qualities criteria in a multivariable format based upon state vectors and matrices of stability and control derivatives. State-space methods are shown to be entirely compatible with conventional flying qualities analysis. Furthermore, they lead to a definition of the well-known control anticipation parameter that has potential utility in the evaluation of higher-order aircraft/control systems.

Introduction

EXISTING flying qualities criteria are based upon modal characteristics and single-input/single-output transfer functions, and while these parameters are adequate to describe the more important aspects of conventional aircraft response, they tend to mask the multivariable nature of the motion, particularly for "superaugmented" configurations whose dynamic modes are revised by strong feedback control. Nevertheless, scalar criteria have been and continue to be useful. The relationship between existing criteria and proposed criteria must be understood by a wide audience, for the essence of a criterion is not only that it be right but that it serve as a common basis for evaluation and decision-making.

With these thoughts in mind, the objective of this paper is to interpret existing criteria in a multivariable format based upon state vectors and matrices of stability and control derivatives. Implicitly or explicitly, stability-and-control engineers have been working with vectors and matrices for decades.[1] The state-space approach is little more than a method of accounting, and it is equally applicable to time-domain (differential equation) or frequency-domain (Fourier/Laplace transform) models of dynamic systems (for example, Ref. 2).

There are at least four advantages to using state-space models. The first is that a large number of scalar equations can be expressed as a small number of vector-matrix equations; hence, the dynamic system can be described in concise yet rigorous fashion. The second is that the vector-matrix expression is independent of the dimension of the multivariable system. Consequently, it is easy to extend simple (low-order) concepts to complex (high-order) systems. The third is that the linearization of the aircraft equations of motion, which are inherently nonlinear, is straightforward. There is no ambiguity regarding what is the nominal and what is the perturbation, and potentially difficult coupling due to non-straight-and-level flight is handled readily. The fourth is that appropriate reduced-order models, including equivalent systems, can be generated easily. The procedures for obtaining these models are relatively transparent in the state-space format.

A Few Basic Rules

The state vector, $x(t)$, contains the aircraft's motion variables, e.g., the four longitudinal variables, axial velocity, normal velocity, pitch rate, and pitch angle:

$$x(t)=\begin{bmatrix} u(t) \\ w(t) \\ q(t) \\ \theta(t) \end{bmatrix} \tag{1}$$

In this example, the control vector, $\delta(t)$, may contain throttle position, flap setting, and elevator displacement:

$$\delta(t)=\begin{bmatrix} \delta T(t) \\ \delta F(t) \\ \delta E(t) \end{bmatrix} \tag{2}$$

Neglecting disturbance inputs, the aircraft equations of motion can be written as a single, nonlinear vector differential equation of appropriate dimension:

$$\dot{x}(t)=f[x(t),\delta(t)] \tag{3}$$

The state, $x(t)$, may be observed as the output of the dynamic system, but other quantities may be observed as well. For example, the outputs may be in a different reference frame (e.g., body-axis dynamic equations with stability-axis outputs) or they may be functions of the state and control (e.g., accelerometer measurements). Consequently, the output vector, $y(t)$, is a (possibly nonlinear) function of the state and control vectors,

$$y(t)=h[x(t),\delta(t)] \tag{4}$$

and its dimension may be larger or smaller than that of the state vector.

The aircraft is in equilibrium if the states are unchanging, or $\dot{x}(t)=0$ [Eq. (3)]. Denoting this "trimmed" condition by a zero subscript, the states and controls for trim are defined by

$$0=f[x_0(t),\delta_0(t)] \tag{5}$$

Perturbations from trim can be characterized by a linear model, which is obtained by a Taylor series expansion of Eq. (3) (with $\dot{x}_0=0$). Using $\Delta(\ )$ to denote the perturbation variables, the linearized dynamic equation is

$$\Delta\dot{x}(t)=F\Delta x(t)+G\Delta\delta(t) \tag{6}$$

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and the linearized output equation is

$$\Delta y(t) = H_x \Delta x(t) + H_\delta \Delta\delta(t) \tag{7}$$

where F, G, H_x, and H_δ are "Jacobian" matrices of partial derivatives evaluated at the trim condition. F contains the aircraft's stability derivatives plus gravitational and kinematic terms, G contains the aircraft's control derivatives, and H_x and H_δ depend on the definition of the output. All of the elements of these matrices are real, scalar variables. With the assumption that the aircraft is trimmed, it can be assumed that the matrices are constant.

The initial response to control is a state rate, which, from Eq. (4) with $\Delta x(0) = 0$ is,

$$\Delta\dot{x}(0) = G\Delta\delta(0) \tag{8}$$

The corresponding initial output rate is found by substituting this in Eq. (7).

Constant perturbations in the control lead to retrimming at a new state. Denoting these constant perturbations by ( )*, the new trim condition is approximated by Eq. (6) with $\Delta\dot{x}(t) = 0$.

$$0 = F\Delta x^* + G\Delta\delta^* \tag{9a}$$

$$F\Delta x^* = -G\Delta\delta^* \tag{9b}$$

Assuming that F is a nonsingular matrix, that is, its inverse exists,

$$\Delta x^* = -F^{-1} G\Delta\delta^* \tag{10}$$

From Eq. (7), the corresponding steady-state output perturbation is

$$\Delta y^* = (-H_x F^{-1} G + H_\delta)\Delta\delta^* \tag{11}$$

The matrices multiplying $\Delta\delta^*$ in Eqs. (10) and (11) represent all the "dc" (zero frequency) gains of the state and output transfer functions described below.

The transfer function matrices are derived from Eqs. (6) and (7). Neglecting initial conditions, the Laplace transform of the linear, constant-coefficient vector dynamic equation is

$$s\Delta x(s) = F\Delta x(s) + G\Delta\delta(s) \tag{12a}$$

or

$$\Delta x(s) = (sI - F)^{-1} G\Delta\delta(s) \tag{12b}$$

$(sI-F)^{-1}G$ is the $(n \times m)$ transfer functrion matrix relating the m controls to the n states, and it can be written as

$$(sI-F)^{-1}G = \frac{\mathrm{Adj}(sI-F)G}{|sI-F|} = \frac{N_x(s)}{\mathrm{d}(s)} \tag{13}$$

where $\mathrm{d}(s)$ is the scalar characteristic polynomial whose factored form contains the n poles (or eigenvalues) of F. $N_x(s)$ is the $(n \times m)$ numerator matrix containing the zeros of the system's state transfer functions.

The output transfer function matrix, $\mathcal{H}(s)$, is found by taking the Laplace transform of Eq. (7) and substituting Eq. (12b) for $\Delta x(s)$. Then

$$\mathcal{H}(s) = H_x (sI-F)^{-1} G + H_\delta = \frac{N_y(s)}{\mathrm{d}(s)} \tag{14}$$

Reduced-order modeling simplifies the analysis of systems with fast and slow dynamic modes. Suppose that fast and slow state variables, Δx_F and Δx_S, can be identified in Eq. (6); then, the equation is partitioned accordingly:

$$\begin{bmatrix} \Delta\dot{x}_S \\ \Delta\dot{x}_F \end{bmatrix} = \begin{bmatrix} F_S & F_F^S \\ F_S^F & F_F \end{bmatrix} \begin{bmatrix} \Delta x_S \\ \Delta x_F \end{bmatrix} + \begin{bmatrix} G_S \\ G_F \end{bmatrix} \Delta\delta \tag{15}$$

If there is little interaction between the fast and slow modes, i.e., if F_S^F and F_F^S are negligible, then Eq. (15) can be truncated, and the motions can be considered separately,

$$\Delta\dot{x}_S \simeq F_S \Delta x_S + G_S \Delta\delta \tag{16}$$

$$\Delta\dot{x}_F \simeq F_F \Delta x_F + G_F \Delta\delta \tag{17}$$

With moderate interaction, the slow modes tend to be modified by the fast modes but not the reverse, and Eq. (15) can be residualized. On the time scale of the fast modes, the effects of the slow modes are negligible, so Eq. (17) can be considered to be an adequate descriptor of the fast modes. On the slow time scale, stable fast modes appear to be instantaneous—so quick that they are always in steady state. To approximate the effects of the fast modes on the slow modes, it is assumed that $\Delta\dot{x}_F = 0$, allowing the lower half of Eq. (15) to provide an algebraic solution of Δx_F in terms of Δx_S and $\Delta\delta$:

$$\Delta x_F = -F_F^{-1}(F_S^F \Delta x_S + G_F \Delta\delta) \tag{18}$$

Substituting Eq. (18) in the remaining solution for Δx_S [Eq. (15)],

$$\Delta\dot{x}_S = (F_S - F_F^S F_F^{-1} F_S^F)\Delta x_S + (G_S - F_F^S F_F^{-1} G_F)\Delta\delta \tag{19}$$

The assumption of instantaneous fast modes can be relaxed. The dynamic lags associated with these modes can be modeled as a single effective time delay, τ, or rise time, T, that is independent of the slow modes. With an equivalent time delay,

$$\Delta x_F(s) = \Delta x_F^* e^{-\tau s} \tag{20}$$

$$\Delta x_S(s) = [sI - (F_S - F_F^S F_F^{-1} F_S^F e^{-\tau s})]^{-1} \times [G_S - F_F^S F^{-1} G_F e^{-\tau s}]\Delta\delta(s) \tag{21}$$

while with an equivalent rise time (to 95% of final value),

$$\Delta x_F(s) = \frac{\Delta x_F^*}{(Ts/3+1)} \tag{22}$$

$$\Delta x_S(s) = \left\{ sI - \left[\frac{F_S - F_F^S F_F^{-1} F_S^F}{(Ts/3+1)} \right] \right\}^{-1} \times \left\{ G_S - \left[\frac{F_F^S F_F^{-1} G_F}{(Ts/3+1)} \right] \right\} \Delta\delta(s) \tag{23}$$

If there is strong interaction between the modes, neither truncation nor residualization is appropriate.

Longitudinal Equations of Motion

Alternate State Vectors

Three definitions of the longitudinal state vector are of particular interest:

$$x_1 = [u\ w\ q\ \theta]^T \tag{24}$$

$$x_2 = [V\ \gamma\ q\ \theta]^T \tag{25}$$

$$x_3 = [V\ \gamma\ q\ \alpha]^T \tag{26}$$

In the first version, the aircraft's translational and angular velocities are expressed in body axes, and the pitch angle relates the body axes to an inertial (or, in this case, Earth-relative) frame. In the second version, the translational velocity is expressed in an Earth-relative frame, with velocity magnitude, V, and flight path angle, γ. In the third case, the vehicle's attitude is measured with respect to the flight path angle rather than an inertial axis. The nonlinear equations which relate u, w, V, α, and γ to each other are

$$V = [u^2 + w^2]^{1/2} \tag{27}$$

$$\alpha = \sin^{-1}(w/V) \tag{28}$$

$$\gamma = \theta - \alpha \tag{29}$$

An exact set of nonlinear equations of motion can be written with any of the alternate state vectors. When body-axis translational velocities are used, it is convenient to express forces in terms of axial- and normal-force coefficients (C_N and C_A, or C_X and C_Z), whereas lift and drag (C_L and C_D) are more appropriate for Earth-relative velocities. The pitching moment coefficient, C_m, is the same in all three cases.

For the most part, we are interested in the linearized equations of motion here, but it is useful to consider a particular nonlinear output variable that does not appear in any of the alternate state vectors. The normal acceleration, a_{z_A}, that is sensed at an arbitrary point, (x_A, y_A, z_A), on the airframe (measured relative to the center of gravity) is

$$a_{z_A} = \dot{w} - qu + pv + (-\dot{q} + pr)x_A + (\dot{p} + qr)y_A - (p^2 + q^2)z_A - g\cos\phi\cos\theta \tag{30}$$

Choosing A to be the pilot's station provides results of particular significance in longitudinal flying qualities. For symmetric flight and at points in the aircraft's plane of symmetry, the normal acceleration is

$$a_{z_A} = \dot{w} - qu - \dot{q}x_A - q^2 z_A - g\cos\theta \tag{31}$$

If it is further assumed that $q^2_{z_A}$ is negligible, the perceived load factor, n_z, (in g units with "up" taken as the positive direction) is

$$n_{z_A} = -\frac{1}{g}(\dot{w} - qu - \dot{q}x_A) + \cos\theta \tag{32}$$

Neglecting unsteady aerodynamic effects, the $\boldsymbol{F}$ and $\boldsymbol{G}$ matrices associated with the third state vector [Eq. (25)] can be expressed as follows:

$$\boldsymbol{F}_3 = \begin{bmatrix} TD_V & -g\cos\gamma_0 & TD_q & TD_\alpha \\ L_V/V_0 & (g/V_0)\sin\gamma_0 & L_q/V_0 & L_\alpha/V_0 \\ M_V & 0 & M_q & M_\alpha \\ -L_V/V_0 & -(g/V_0)\sin\gamma_0 & (1 - L_q/V_0) & -L_\alpha/V_0 \end{bmatrix} \tag{33}$$

$$\boldsymbol{G}_3 = \begin{bmatrix} TD_{\delta T} & TD_{\delta F} & TD_{\delta E} \\ L_{\delta T}/V_0 & L_{\delta F}/V_0 & L_{\delta E}/V_0 \\ M_{\delta T} & M_{\delta F} & M_{\delta E} \\ -L_{\delta T}/V_0 & -L_{\delta F}/V_0 & -L_{\delta E}/V_0 \end{bmatrix} \tag{34}$$

The stability derivatives have conventional definitions; TD_V represents the forward specific force sensitivity to velocity perturbations $(T_V - D_V)$, and so on.[1]

$\boldsymbol{F}_3$ provides the best description of aircraft motions for CTOL flying qualities investigations because it is the most nearly block diagonal in its basic form. In other words, $\boldsymbol{F}_3$ can be partitioned in the following way:

Phugoid parameters	Short period-to-phugoid coupling
Phugoid-to-short period coupling	Short period parameters

The truncated phugoid and short period models, based on the upper left and lower right (2×2) submatrices of $\boldsymbol{F}_3$ make the case. The respective characteristic equations are

$$|s\boldsymbol{I} - \boldsymbol{F}_P| = \begin{vmatrix} (s - TD_V) & g\cos\gamma_0 \\ -L_V/V_0 & [s - (g/V_0)\sin\gamma_0] \end{vmatrix}$$

$$= s^2 - [TD_V + (g/V_0)\sin\gamma_0]s + (g/V_0)(L_V\cos\gamma_0 + TD_V\sin\gamma_0) = 0 \tag{35}$$

$$|s\boldsymbol{I} - \boldsymbol{F}_{SP}| = \begin{vmatrix} (s - M_q) & -M_\alpha \\ (L_q/V_0 - 1) & (s + L_\alpha/V_0) \end{vmatrix}$$

$$= s^2 + (L_\alpha/V_0 - M_q)s + [M_\alpha(L_q/V_0 - 1) - M_q L_\alpha/V_0] = 0 \tag{36}$$

These equations possess the conventional reduced-order approximations for phugoid and short period natural frequencies and damping ratios. In level flight,

$$\omega_{n_p} \simeq [gL_V/V_0]^{1/2} \tag{37}$$

$$\zeta_P \simeq -TD_V/2\omega_{n_P} \tag{38}$$

$$\omega_{n_{SP}} \simeq [-M_\alpha(1 - L_q/V_0) - M_q L_\alpha/V_0]^{1/2} \tag{39}$$

$$\zeta_{SP} \simeq (L_\alpha/V_0 - M_q)/2\omega_{n_{SP}} \tag{40}$$

Load factor perturbations from trim also will be of interest. Expanding Eq. (32) in a Taylor series and retaining only the linear term,

$$\Delta n_{z_A} = -\frac{1}{g}(\Delta\dot{w} - u_0\Delta q - x_A\Delta\dot{q}) - \sin\theta_0\Delta\theta$$

$$= -\frac{1}{g}(V_0\cos\alpha_0\Delta\dot{\alpha} + \sin\alpha_0\Delta\dot{V} - V_0\cos\alpha_0\Delta q - x_A\Delta\dot{q}) - \sin(\gamma_0 + \alpha_0)(\Delta\gamma + \Delta\alpha) \tag{41}$$

Truncated Short Period Model

From the above it can be seen that a second-order short period model considering only elevator control and neglecting L_q/V_0 could be written as

$$\Delta\dot{\boldsymbol{x}}_{SP}(t) = \boldsymbol{F}_{SP}\Delta\boldsymbol{x}_{SP}(t) + \boldsymbol{G}_{SP}\Delta\delta E(t) \tag{42a}$$

or

$$\begin{bmatrix} \Delta\dot{q}(t) \\ \Delta\dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} M_q & M_\alpha \\ 1 & -L_\alpha/V_0 \end{bmatrix} \begin{bmatrix} \Delta q(t) \\ \Delta\alpha(t) \end{bmatrix} + \begin{bmatrix} M_{\delta E} \\ -L_{\delta E}/V_0 \end{bmatrix} \Delta\delta E(t) \tag{42b}$$

The Laplace transforms of Eq. (42) are

$$\Delta x_{SP}(s) = (sI - F_{SP})^{-1} G_{SP} \Delta\delta E(s) \tag{43}$$

and

$$\begin{bmatrix} \Delta q(s) \\ \Delta\alpha(s) \end{bmatrix} = \frac{\begin{bmatrix} (s+L_\alpha/V_0) & M_\alpha \\ 1 & (s-M_q) \end{bmatrix} \begin{bmatrix} M_{\delta E} \\ -L_{\delta E}/V_0 \end{bmatrix} \Delta\delta E(s)}{s^2 + (L_\alpha/V_0 - M_q)s - (M_\alpha + M_q L_\alpha/V_0)} \tag{44}$$

The latter leads to the two scalar transfer functions,

$$\frac{\Delta q(s)}{\Delta\delta E(s)} = \frac{M_{\delta E}[s + (L_\alpha - M_\alpha L_{\delta E}/M_{\delta E})/V_0]}{s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega^2_{n_{SP}}} \tag{45}$$

$$\frac{\Delta\alpha(s)}{\Delta\delta E(s)} = \frac{(-L_{\delta E}/V_0)[s - (M_q + V_0 M_{\delta E}/L_{\delta E})]}{s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega^2_{n_{SP}}} \tag{46}$$

The associated normal load factor transfer function is derived from Eqs. (41) and (14). For the reduced-order model, that is, neglecting ΔV and $\Delta\gamma$, Eq. (41) is

$$\Delta n_{z_A} = -\frac{1}{g}[V_0 \cos\alpha_0 (\Delta\dot{\alpha} - \Delta q) - x_A \Delta\dot{q}] - \sin(\gamma_0 + \alpha_0)\Delta\alpha \tag{47}$$

Using Eq. (42b) to eliminate Δq and $\Delta\alpha$,

$$\Delta n_{Z_A} = (x_A M_q/g)\Delta q + [(L_\alpha \cos\alpha_0 + x_A M_\alpha)/g - \sin(\gamma_0+\alpha_0)]\Delta\alpha + [(L_{\delta_E}\cos\alpha_0 + x_A M_{\delta E})/g]\Delta\delta E \tag{48}$$

Defining Δn_{Z_A} as the output "vector" (actually a scalar), the corresponding H_x and H_δ are contained in Eq. (48). The $\Delta n_{Z_A}(s)/\Delta\delta E(s)$ transfer function can then be found from Eq. (14); however, in the present case, Eqs. (45), (46), and (48) can be combined readily to form

$$\frac{\Delta n_{Z_A}(s)}{\Delta\delta E(s)} = \{(x_A M_q/g) N^q_{\delta E}(s) + [(L_\alpha \cos\alpha_0 + x_A M_\alpha)/g - \sin(\gamma_0+\alpha_0)] N^\alpha_{\delta E}(s) + [(L_{\delta E}\cos\alpha_0 + x_A M_{\delta E})/g] \mathrm{d}(s)\}/\mathrm{d}(s) \tag{49}$$

The pitch rate and angle-of-attack transfer functions have first-order numerators, whereas the normal load factor transfer function has a second-order numerator arising from $\mathrm{d}(s)$; hence, the terminology of Ref. 1 can be used to express Eqs. (45), (46), and (49) as

$$\frac{\Delta q(s)}{\Delta\delta E(s)} = \frac{K_q(s+1/T_{\theta 2})}{(s^2+2\zeta\omega_n s+\omega_n^2)_{SP}} \tag{50}$$

$$\frac{\Delta\alpha(s)}{\Delta\delta E(s)} = \frac{K_\alpha(s+1/T_{w1})}{(s^2+2\zeta\omega_n s+\omega_n^2)_{SP}} \tag{51}$$

$$\frac{\Delta n_{Z\alpha}(\lambda)}{\Delta\delta E(s)} = \frac{K_{nZ}(s+1/T_{h2})(s+1/T_{h3})}{(s^2+2\zeta\omega_n s+\omega_n^2)_{SP}} \tag{52}$$

Additional Short Period Flying Qualities Parameters

Two parameters of particular interest can be derived from the short period model. The control anticipation parameter (CAP) relates initial pitch acceleration to "steady-state" normal load factor,[3] and it forms the basis of the primary military specification on short period flying qualities.[4] The C^* criterion provides a step-response envelope within whch the derived variable C^* presumably must lie to provide satisfactory flying qualities.[5] CAP and C^* are easily defined using the state-space approach.

In the usage of this paper, CAP is defined for a constant elevator input, $\Delta\delta E^*$, as

$$\mathrm{CAP} = \Delta\dot{q}(0)/\Delta n^*_{Z_A} \tag{53}$$

From Eq. (8), the CAP numerator is

$$\Delta\dot{q}(0) = M_{\delta E}\Delta\delta E^* \tag{54}$$

while from Eq. (48), the denominator is

$$\Delta n^*_{Z_A} = (x_A M_q/g)\Delta q^* + [(L_\alpha\cos\alpha_0 + x_A M_\alpha)/g - \sin(\gamma_0+\alpha_0)]\Delta\alpha^* + [(L_{\delta E}\cos\alpha_0 + x_A M_{\delta E})/g]\Delta\delta E^* \tag{55}$$

The constant values, Δq^* and $\Delta\alpha^*$, that correspond to $\Delta\delta E^*$ are found from Eq. (10). Using Eq. (10) and the truncated short period model, this yields

$$\begin{bmatrix} \Delta q^* \\ \Delta\alpha^* \end{bmatrix} = \frac{\begin{bmatrix} (-M_{\delta E}L_\alpha/V_0 + M_\alpha L_{\delta E}/V_0) \\ (-M_{\delta E} - M_q L_{\delta E}/V_0) \end{bmatrix}}{(M_\alpha + M_q L_\alpha/V_0)} \Delta\delta E^* \tag{56}$$

These values are substituted in Eq. (55) and used, along with Eq. (54), to define CAP.

This is a fairly general expression; if Δn_Z is measured at the center of gravity ($x_A = 0$) and if $L_{\delta E} = \gamma_0 = \alpha_0 = 0$, Eq. (53) reduces to

$$\mathrm{CAP} = \frac{-(M_\alpha + M_q L_\alpha/V_0)}{(L_\alpha/g)} \tag{57a}$$

$$\triangleq \frac{\omega^2_{n_{SP}}}{(n_Z/\alpha)} \tag{57b}$$

The latter form, an approximation to Eq. (53), is used in the flying qualities specification.[4]

The parameter C^* is defined as

$$C^*(t) = \Delta n_{Z_{\mathrm{PILOT}}}(t) + (V_c/g)\Delta q(t) \tag{58}$$

where V_c is an arbitrary "crossover" velocity, typically defined as 122 m/s (400 ft/s). Below the crossover velocity, $C^*(t)$ is dominated by pitch rate response, while above it primarily reflects normal acceleration response. C^* can be defined as an element of the output vector [Eq. (7)], choosing x_A to represent the pilot's station.

The limitations of the truncated short period model should be understood. The model is useful only if the short period and phugoid modes are not strongly coupled. Even then the so-called "steady-state" characteristics of the model are not precise because the phugoid mode is neglected. The truncated short period "steady state" is well below the phugoid natural frequency, violating the assumptions of truncation. For example, if Eqs. (10) and (11) are applied to the fourth-order model [Eq. (33) and (34)], Δq^* and Δn^*_Z are found to be zero, a fact that is corroborated by frequency response examples in Ref. 1. Hence, "steady state," as used in the definition of CAP and C^* step response, really is "quasi-steady state." It may be a useful definition on the short period time scale, but it is wholly inadequate on the phugoid time scale. (More discussion of equilibrium response can be found in Ref. 6.)

A further qualification is that nonrigid-body modes of motion affect the short period mode, as recognized by the development of longitudinal equivalent systems.[7] An alternate to frequency-response definitions of equivalent systems for higher-order aircraft can be found in residualization, introduced in the previous section, and applied to the phugoid mode below.

Residualized Phugoid Model

Although the short period flying qualities criteria attract the most attention, a number of specifications are placed on the phugoid mode and its associated variables (ΔV and $\Delta\gamma$). It is possible to build a discussion of the phugoid mode on a truncated model such as the one used for the short period mode; however, it is hardly more difficult and somewhat more accurate if the second-order phugoid model accounts for the short period (attitude) dynamics. With adequate separation of the time scales, residualization assumes that short period transients have decayed, leaving only quasi-steady-state effects on the phugoid mode.

As noted previously, F_3 [Eq. (33)] can be partitioned into (2×2) matrices containing phugoid parameters, F_P; short period parameters, F_{SP}; and coupling parameters, F_{SP}^P and F_P^{SP}. The phugoid is slow, the short period is fast, and Eqs. (14-18) are applied to the problem. To summarize, with $\gamma_0 = TD_q = L_q = 0$,

$$F_P = \begin{bmatrix} TD_V & -g \\ L_V/V_0 & 0 \end{bmatrix} \tag{59}$$

$$F_{SP} = \begin{bmatrix} M_q & M_\alpha \\ 1 & -L_\alpha/V_0 \end{bmatrix} \tag{60}$$

$$F_P^{SP} = \begin{bmatrix} M_V & 0 \\ -L_V/V_0 & 0 \end{bmatrix} \tag{61}$$

$$F_{SP}^P = \begin{bmatrix} 0 & TD_\alpha \\ 0 & L_\alpha/V_0 \end{bmatrix} \tag{62}$$

Denoting the residualized second-order phugoid stability and control matrices by primes, Eq. (19) indicates that

$$F_P' = F_P - F_{SP}^P F_{SP}^{-1} F_P^{SP}$$

$$= \begin{bmatrix} \left[TD_V + TD_\alpha \dfrac{(M_V + M_q L_V/V_0)}{\alpha(M_\alpha + M_q L_\alpha/V_0)}\right] & -g \\ \left[L_V/V_0 + (L_\alpha/V_0)\dfrac{(M_V + M_q L_V/V_0)}{(M_\alpha + M_q L_\alpha/V_0)}\right] & 0 \end{bmatrix} \tag{63}$$

$$G_P' = G_P - F_{SP}^P F_{SP}^{-1} G_{SP} \tag{64}$$

Although throttle and flap settings have major effects on the phugoid mode, only elevator effects are considered here,

$$G_P = \begin{bmatrix} TD_{\delta E} \\ L_{\delta E}/V_0 \end{bmatrix} \tag{65}$$

$$G_{SP} = \begin{bmatrix} M_{\delta E} \\ -L_{\delta E}/V_0 \end{bmatrix} \tag{66}$$

Equation (64) becomes

$$G_P' = \begin{bmatrix} \left[TD_{\delta E} + TD_\alpha \dfrac{(M_{\delta E} + M_q L_{\delta E}/V_0)}{(M_\alpha + M_q L_\alpha/V_0)}\right] \\ \left[L_{\delta E}/V_0 + (L_\alpha/V_0)\dfrac{(M_{\delta E} + M_q L_{\delta E}/V_0)}{(M_\alpha + M_q L_\alpha/V_0)}\right] \end{bmatrix} \tag{67}$$

The flying qualities specification[4] places separate requirements on phugoid stability, flight path stability, and speed stability. For the first condition, phugoid damping ratio can be estimated from $|sI - F_P|$ in the usual manner. The second condition refers to the amount of flight path angle change which occurs when airspeed is controlled by elevator alone. Because the steady-state effect of elevator on airspeed and flight path angle, including short period effects, can be found by substituting Eqs. (63) and (67) in Eq. (10),

$$\begin{bmatrix} \Delta V^* \\ \Delta\gamma^* \end{bmatrix} = -(F_P')^{-1} G_P' \Delta\delta E^* \tag{68}$$

the desired relationship, $\Delta\gamma^*/\Delta V^*$, is calculated easily. Similarly, the control position gradient which characterizes speed stability is simply $\Delta\delta E^*/\Delta V^*$.

Contrary to the implication of the specification that altitude must remain constant, nonzero $\Delta\delta E^*$ normally would cause nonzero $\Delta\gamma^*$ on the phugoid time scale unless throttle or flap setting is altered. The proviso is significant, because on a much longer time scale, atmopsheric density gradient effects, which can be incorporated in a third-order phugoid/height mode model, would force $\Delta\gamma^*$ to zero in the limit.

Use of Pitch Angle as a Velocity/Flight Path Control

It is often said that the phugoid mode cannot exist when pitch angle is closely controlled. A better way of putting it is that the phugoid mode's two roots are virtually certain to be real. This is easily seen by considering the second state vector [Eq. (25)] and its associated fundamental matrix. Assuming that pitch angle is directly controllable and that $TD_q = L_q = \gamma_0 = 0$, the resulting truncated (ΔV,$\Delta\gamma$) dynamic model is

$$\begin{bmatrix} \Delta\dot V \\ \Delta\dot\gamma \end{bmatrix} = \begin{bmatrix} TD_V & -(TD_\alpha + g) \\ L_V/V_0 & -L_\alpha/V_0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix} + \begin{bmatrix} TD_\alpha \\ L_\alpha/V \end{bmatrix} \Delta\theta \tag{69}$$

$$\triangleq F_{P_2} \Delta x_P + G_{P_2} \Delta\theta$$

The roots of $|sI - F_{P_2}|$ are

$$\lambda_{1,2} = \frac{-(L_\alpha/V_0 - TD_V)}{2}$$
$$\pm\sqrt{\frac{(L_\alpha/V_0 - TD_V)^2}{4} - [(L_\alpha/V_0)(TD_\alpha + g) - TD_V L_\alpha/V_0]} \tag{70}$$

which would be real and negative under normal conditions. With negligible drag effects, the roots would be 0 and $-L_\alpha/V_0$, respectively. The $\Delta\gamma$ root, $-L_\alpha/V_0$, happens to be near the zero of $\Delta q(s)/\Delta\delta E(s)$ [Eq. (45)], but, of course, it arises for entirely different reasons. The $\Delta V(s)/\Delta\theta(s)$ and $\Delta\gamma(s)/\Delta\theta(s)$ transfer functions are found by applying Eq. (12) to Eq. (69).

Further Considerations

Prefilters and Time Delays

Higher-order aircraft/control systems are difficult to evaluate not only as a consequence of increased technical complexity but because flying qualities specifications for such systems have yet to be defined. Attention is directed to establishing models for higher-order systems that are charcterized by constant dynamic parameters and a single pilot control input: longitudinal ("pitch") stick, δS. The previous sections referred to the elevator, δE, as the control, under the usual assumption that δE could be related to the cockpit controller by linear gearing and/or force gradient factors containing no compensation or delay. The assumption of linearity is retained, and some effects of dynamic elements between the pilot's input and the elevator's motion are discussed below.

The Laplace transform of the elevator position is defined as a dynamically altered function of the stick position,

$$\Delta\delta E(s) = P(s)\Delta\delta S(s) \tag{71}$$

where $P(s)$ is a prefilter function describing control dynamics, e.g., a pure time delay, $e^{-\tau s}$, or an assemblage of poles and zeros. In some analyses, it may be desirable to replace the pure time delay by its first-order Padé approximation,[8]

$$e^{-\tau s} = \frac{-(s-2/\tau)}{(s+2/\tau)} \tag{72}$$

The Padé approximation for a time delay contains a non-minimum-phase zero, which reverses the sense (or "direction") of initial response to step inputs. The command prefilter (or delay) has an identical effect on all of the scalar transfer functions of the system: an equivalent command delay has the same effect on $\Delta q(s)/\Delta\delta S(s)$ as on $\Delta n_Z(s)/\Delta\delta S(s)$. Alternatively, a rational $P(s)$ can be added to the state-space model by enlarging the state vector to include an additional component for each pole of $P(s)$.[2,9]

The augmented system clearly is of higher order than the original system; however, if the prefiltering elements have higher bandwidth, residualiztion can be applied to the problem. Because Δx does not "feed back" to the command prefilter, the rigid-body modes are unaffected by the approximation, but the control effect is modified. The residualized command prefilter shifts the zeros but not the poles of the low-order equivalent transfer functions.

Multivariable analysis becomes particularly important if the compensation elements are in the range of the short period natural frequency or below, because mode shapes, as portrayed by the aircraft's eigenvectors (or normal mode vectors[10]), may be very unnatural. The resulting response is not necessarily bad—it is just not well represented by accepted definitions of natural aircraft modes.

CAP Revisited

It can be seen from Eq. (53) that the basic definition of CAP is response-centered, depending on initial pitch acceleration and quasi-steady-state load factor. The reasoning behind this criterion is that the pitch acceleration sensed by the pilot's inner ear should be in harmony with the steady-state load factor that results from pitch control.[3] The commonly used approximation [Eq. (57)] is parameter-centered, and this is frequently misinterpreted as implicit justification for parameter-centered criteria in lieu of response-centered criteria.

These definitions would be numerically identical if the actual system were accurately portrayed by the simplified system,[7,11] that is, measurements of $\Delta\dot{q}(0)$ and $\Delta n^*_{Z_A}$ could be used to calculate the CAP that is to be compared with the short period specification.[4] This suggests an alternative for flying qualities evaluation of higher-order systems that retains the spirit of the currently defined equivalent systems approach,[7] but which may be substantially simpler to implement with both numerical (simulated) and flight test data: define CAP by values of $\Delta\dot{q}(0)$ and $\Delta n^*_{Z_A}$ rather than equivalent values of ω^2_{nSP} and n_z/α. This step-response approach bears resemblance (but is not identical) to the C^* criterion. Choosing a value of $\Delta q(0)$ could be a problem, as higher-order systems are often characterized by initially delayed response. A solution can be found by defining an equivalent $\Delta\dot{q}(0)$, leading to the equivalent control anticipation parameter (CAP′).

Two approximations can be considered. The initial pitch acceleration can be approximated by

$$\Delta\dot{q}(0) \simeq \Delta q^*/\Delta t_{q^*} \tag{73}$$

where Δt_{q^*} is the equivalent rise time, i.e., the time required for Δq to first reach its quasi-steady-state value, or a linear-quadratic buildup in pitch rate could be assumed, as shown in Fig. 1. The figure illustrates three higher-order step responses that would yield the same Δt_{q^*}. Because the equivalent $\Delta\dot{q}(0)$ often would overestimate the initial $\Delta\dot{q}(0)$, satisfactory values of CAP′ may be larger than the corresponding values of CAP. As a practical matter, it could be desirable to measure aircraft rise time to reach 90 or 95% of the quasi-steady-state pitch rate, thus allowing overdamped as well as underdamped systems to be covered by the same flying qualities criteria; however, the corresponding CAP′ then would not be identical to the original definition.

In application, Δt_{q^*} contains all factors which contribute to the delay in achieving pitch rate response, including short period rise time, control system filtering, actuator lags, pure delays and sampling delays (i.e., half the sampling interval), and other higher-order effects. Systems with computation delays and sampling effects, e.g., digital flight control systems, can be evaluated in the same way that analog systems are evaluated.† CAP′ is defined as

$$\text{CAP}' = \Delta q^*/\Delta t_{q^*}\Delta n^*_{z_A} \tag{74}$$

To the same level of approximation used in Eq. (72), this can be expressed as

$$\text{CAP}' = \frac{(g/V_0)}{\Delta t_{q^*}} \tag{75}$$

implying that if CAP′ is the controlling parameter, equivalent rise time is the only factor other than airspeed that is of concern to the pilot. Conversely, for a given value of CAP′, the corresponding equivalent rise time is

$$\Delta t_{q^*} = \frac{(g/V_0)}{\text{CAP}'} \tag{76}$$

The limiting rise time associated with Eq. (74) is

$$\Delta t_{q^*} = \frac{\Delta q^*}{(\text{CAP}')\Delta n^*_{z_A}} \tag{77}$$

where A could be chosen as the pilot's station.

The Δt_{q^*} boundaries shown in Fig. 2 correspond to the short period frequency and acceleration sensitivity boundaries of the current flying qualities specification.[4] The figure presents the lines of constant CAP′ [Eq. (76)] associated with rapid maneuvering (flight phase A) and takeoff and landing maneuvers (flight phase C) for three levels of flying qualities. Level 1 is "clearly adequate for the mission," while levels 2 and 3 reflect increased pilot workload or degradation in mission effectiveness. The boundaries between these levels are loosely equivalent to pilot opinion ratings[2] of 3.5 and 6.5, respectively. The current specification prescribes additional limiting values as functions of aircraft class at low ω_{nSP} and n_Z/α, and it would be appropriate to develop similar limitations on maximum allowable Δt_{q^*} at low airspeeds.

†It should be noted, however, that experienced pilots are sensitive to subtle distinctions in delay type; Ref. 13 provides evidence that pilots are better able to adapt to pure delay than to equivalent sampling delay.

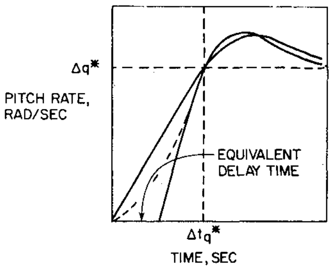

Fig. 1 Definition of equivalent rise time, Δt_q^*.

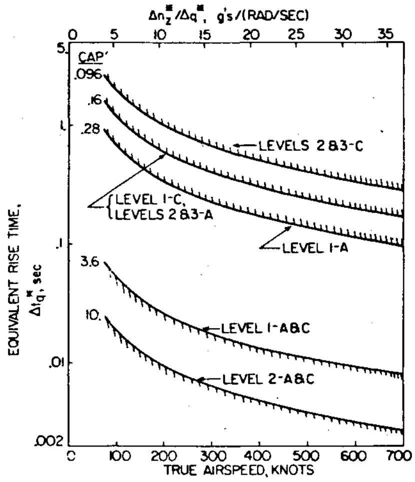

Fig. 2 Example of short period flying qualities boundaries based on equivalent rise time, Δt_q^*.

Table 1 Boundary values of pitch-rate rise time

Flight phase	Δt_{q^*}, ms	V, knots
A (two pilots)	380	105
B	440	105
C (precision approach)	290	86
C (precision approach)	275	105
C (conventional landing)	485	75

Furthermore, flight phase C criteria need not be specified for airspeeds beyond those normally encountered during takeoff and landing. For aircraft with conventional controls, true airspeed can be considered the independent variable in Fig. 2. For aircraft with direct lift control enhancement of pitch response, the $\Delta n_z^* / \Delta q^*$ scale is more appropriate.

Some indication of suitable low-speed Δt_{q^*} boundaries can be gained from the flight evaluations reported in Ref. 13. The tests were flown with Princeton University's Variable-Response Research Aircraft (VRA), which is equipped with a microprocessor-based digital flight control system (Micro-DFCS). The VRA was flown at three airspeeds (75, 86, and 105 KIAS) with digital sampling rates of 2-20 per second and pure time delays, τ, of 5-405 ms. These sampling rates correspond to sampling intervals, T, of 500-50 ms. (The effective sampling delay is $T/2$.) Elevator rise time to 100% deflection, Δt_E, was approximately 35 ms, while the aircraft's speed-dependent rise time to 100% Δq^*, Δt_A, was 246, 238, and 198 ms (75, 86, and 105 KIAS). The equivalent rise time could be calculated as

$$\Delta t_q^* = \Delta t_A + \Delta t_E + \tau + T/2 \tag{78}$$

and it ranged 263-736 ms in these tests.‡ The pilot opinion ratings reported in Ref. 13 have been plotted against the new parameter, Δt_{q^*}. Ratings of 3.5, the boundary between "satisfactory" flying qualities, and those whose "deficiencies warrant improvement" are obtained for the values of Δt_{q^*} shown in Table 1.

The preliminary findings of Table 1 except were obtained with a single pilot, except as noted. Flight phase B refers to flight path control at altitude. All these Δt_{q^*} values are well within the level 1 region of Fig. 2, and flight phase C results, which involved tracking an aircraft carrier approach mirror, are more restrictive than the flight phase A result.

Pitch rate rise time is reported to correlate well with pilot opinion of low-speed longitudinal maneuvering characteristics in Ref. 14. The present results are consistent with this finding and provide additional guidance concerning the relationships between rise time, flight speed, and the ratio of pitch rate to normal acceleration response.

‡Delays and dynamic lags are not strictly additive, as implied here. The definition of Eq. (78) is chosen for example only.

Conclusion

State-space methods are entirely compatible with conventional flying qualities analysis, and they provide an improved framework for extending existing single-control techniques to higher-order systems. Relationships between important variables and flying qualities parameters are derived easily, reduced-order models are generated, and the mechanisms for addressing more complex problems such as pilot location effects and phugoid/short period coupling are developed. A time-domain definition of the control anticipation parameter is seen to be numerically identical to the frequency-domain definition, and it is somewhat more amenable to the incorporation of time delays through the mechanism of an equivalent rise time. While the state-space approach can be helpful for current flying qualities studies, perhaps its most significant application is yet to come in the development of flying qualities criteria for the multiple-control case.

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